1. Given a linear space **V** of symmetric 2x2 matrices with real coefficients and with a scalar product between two vectors X and Y in space **V** defined by:

$$(X,Y) = tr\left([1 \quad 1]XY[1 \quad 1]^T\right) + tr(XY), \text{ where } tr(A) \text{ is trace of the matrix}$$

Find an othonormal basis of the space V.

<u>Hint</u>: You can start with one simple possible basis for **V** consisting of linearly independent vectors {a1, a2, a3} given by:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

2.
This questions consists of 3 parts a, b and c:

a) determine the general solution y(x) of the differential equation

$$4y''+28y'+49y=0$$

using a strategy developed by d'Alembert.

b)show that the solutions that constitute the general solution of the ODE are linearly independent and therefore form a fundamental set of solutions.

c) determine the exact solution of the ODE for boundary conditions

$$y(0)=2 \quad \text{and} \quad y'(0)=2/3$$

3.

Compute $3^{1/3}$ to 2 digits of accuracy using Newton's method with the initial value of 1.

(1) Show how you obtain the numerical solution step by step.

(2) Write a Matlab (or Fortran or C) code to implement your numerical method.

4.

a) Find the regions in the *xy* plane where the equation:

$$yu_{xx} - 2u_{xy} + xu_{yy} = 0$$

is hyperbolic, elliptic, and parabolic. Sketch these regions in the *xy* plane.

b) Using the Laplace transform, find an analytical solution to the wave equation,

$$u_{tt} = c^2 u_{xx}$$

for $0 < x < \infty$, with boundary and initial conditions given by,

$$u(0,t) = f(t) \qquad u(x,0) = u_t(x,0) = 0$$

Assume that $u(x,t) \to 0$ as $x \to +\infty$ and express your solution in terms of the function f and other relevant functions. You may use the Laplace transforms properties given below.

Function	**Laplace Transform**
$af(t)+bg(t)$	$aF(s)+bG(s)$
$\dfrac{df}{dt}$	$sF(s)-f(0)$
$\dfrac{d^2 f}{dt^2}$	$s^2F(s)-sf(0)-f'(0)$
$e^{bt}f(t)$	$F(s-b)$
$\dfrac{f(t)}{t}$	$\int_0^\infty F(s')\,ds'$
$tf(t)$	$-\dfrac{dF}{ds}$
$H(t-b)f(t-b)$	$e^{-bs}F(s)$
$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$

* $H(\cdot)$ denotes the Heaviside function given by $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$.

Hint: The general solution to an ODE of the form $\dfrac{d^2 f}{dx^2} = a^2 f$ is given by $f(x) = c_1 e^{-ax} + c_2 e^{ax}$ where c_1 and c_2 are constants.