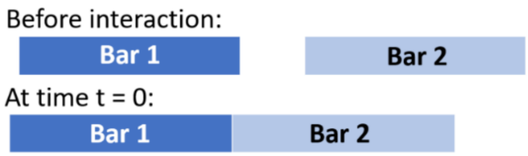
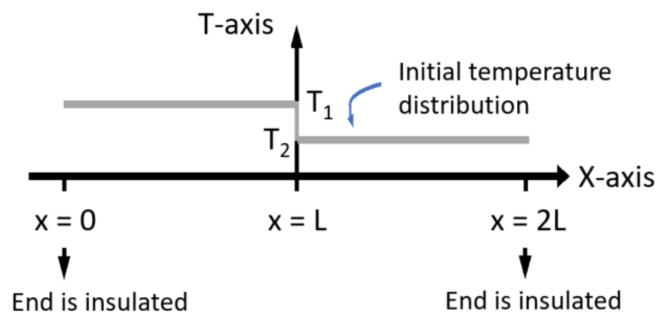
APPLIED MATH WRITTEN EXAM

1. Consider two bars, each initially at constant temperature: T_1 (bar 1) and T_2 (bar 2). The length of each bar is equal to L and cannot change. At time t = 0, the bars are brought together and heat exchange will affect the temperature distribution. Further, the bar ends that are not in contact are isolated/insulated so no heat is added or lost through those ends. At t = 0, the temperature distribution f(x) is described by a Heaviside step function as in the figure and the formula below.



Initial temperature distribution:

$$f(x) = \begin{cases} T_1 & for\ x < L \\ \frac{T_1+T_2}{2} & for\ x = L \\ T_2 & for\ x > L \end{cases}$$



The problem satisfies the below heat conduction equation and with $u(x,t)$ the temperature at any time and place in the joined metal bars as in Eq. (1). Further, because the bar ends not in contact are isolated, the following condition in Eq. (2) holds:

$$A^2 \frac{\partial^2 u}{\partial x \partial x} - \frac{\partial u}{\partial t} = 0, \text{ for every } 0 < x < 2L \text{ and } t > 0 \qquad \text{Eq. (1)}$$

$$\frac{\partial u}{\partial x} = 0 \text{ at x = 0 and x = 2L, for any time } t > 0 \qquad \text{Eq. (2)}$$

Describe mathematically how the temperature in the system consisting of the two coupled bars evolves with time and show the final temperature distribution. Solve the problem until you require the initial condition. From there onwards it is sufficient, (i) for general values of time t to describe in words how the problem would be solved further, and (ii) to give the explicit solution for the final temperature distribution only.

Hint: The cosine Fourier series of the Heaviside function above is:

$$f(x) = \frac{T_1+T_2}{2} + \sum_{n\ odd} \frac{2(T_1-T_2)}{n\pi} cos\left(\frac{n\pi}{2L}(x-L) - \frac{\pi}{2}\right)$$

2. Answer the following questions:

   a. Solve the initial value problem defined by the 4th order linear ODE:

$$y'''' = \frac{d^4 y}{dt^4} = -\sin t + \cos t$$
$$y'''(0) = 10,\ y''(0)\, y'(0) = -1,\ y(0) = 0$$

   b. Find the general solution to the inhomogeneous first order linear ODE:

$$\frac{dy}{dx} + 10xy = x$$

3. For $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$ with the given condition of $y(0)=\left.\frac{dy}{dx}\right|_{x=0}=\left.\frac{d^2y}{dx^2}\right|_{x=0}=1$,

solve for $\left.\frac{dy}{dx}\right|_{x=1.0}$ based on Euler's method with a step size of 0.5.

4. Answer the following questions:

(a) Find the flux of vector field $\boldsymbol{F}(x, y, z) = z\,\boldsymbol{i} + 2y\,\boldsymbol{j} + 10z\,\boldsymbol{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

(b) Evaluate $\iint_S F \cdot dS$ where $\boldsymbol{F}(x, y, z) = xy\,\boldsymbol{i} + \left(y^2 + e^{xz^2}\right)\boldsymbol{j} + \sin(xy)\,\boldsymbol{k}$ and S is the surface of the region E bounded by $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, $y + z = 2$.