1.a) Briefly, discuss the main sources of error in finite difference approximations of PDE's.

2.b) Using only the grid points u_i, u_{i-1}, u_{i-2}, develop a finite difference formula for $(du/dx)\big|_i$ (the first derivative evaluated at grid point x_i) whose error is $O\left(\Delta x^2\right)$. You should use a Taylor series expansion about u_i.

2. Calculate the following

(a) $$div\left[f(r)\boldsymbol{r} \right] = \dots$$

(b) $$\boldsymbol{curl}\left[f(r)\boldsymbol{r} \right] = \dots$$

(c) Using Stokes' theorem, evaluate the following integral over the closed surface S:

$$\iint \boldsymbol{curl}\, \boldsymbol{b} \cdot d\boldsymbol{S} = \dots$$

Here, as customary:

$div(\boldsymbol{vector}) \equiv \nabla \cdot (\boldsymbol{vector})$, $\boldsymbol{curl}(\boldsymbol{vector}) \equiv \nabla \times (\boldsymbol{vector})$,

$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$: position vector,

$f(r)$: arbitrary (smooth) function of r, $r^2 = x^2 + y^2 + z^2$,

$\boldsymbol{b}$: arbitrary (smooth) vector function,

$d\boldsymbol{S} = \boldsymbol{n}\, dS$: vector *area* element, with outward unit normal $\boldsymbol{n}$.

3.a) Find three eigenvalues of the following matrix A, and the corresponding eigenvector matrix, M.

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

3.b) Explain why $A^{1001} = A$. Is $A^{1000} = I$?

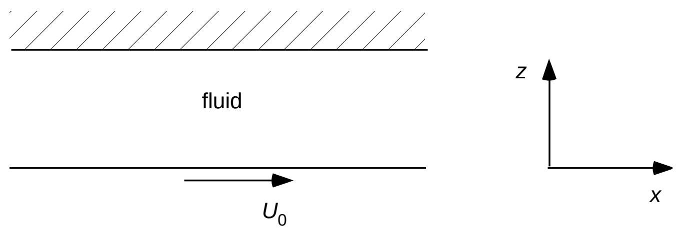
3.c) Find the three diagonal entries of e^{At}. Explain your derivation.

3.d) The matrix $A^T A$, for the same A, is

$$A^T A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{bmatrix}$$

How many eigenvalues of $A^T A$ are positive, zero, or negative? Do not calculate them, but explain your answer.

4. A two-dimensional channel with a fixed upper surface and a movable lower surface contains an incompressible fluid initially at rest. At time $t = 0$, the bottom surface is impulsively given a velocity U_0 in the x-direction as shown in the figure below.



The resulting transient parallel flow in the x-direction is governed by the following *dimensionless* partial differential equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2}, \quad u(0,t) = 1, \quad u(1,t) = 0, \quad u(z,0) = 0 .$$

Determine the transient velocity $u(z,t)$ of the fluid. How would you use this solution to estimate the time it takes for the flow to reach steady state?