<u>Instructions</u>

Please complete all **4** problems attached.

Problem 1: Vector Calculus

Show that

$$\int d\boldsymbol{r} \times \boldsymbol{q} = \iint (d\boldsymbol{S} \times \nabla) \times \boldsymbol{q} \qquad (1)$$

where

$\boldsymbol{q}$: arbitrary ("well behaved") vector field, the integral $\int$ is taken over the *closed curve C* [$\boldsymbol{r}$: position vector defining *C* and $d\boldsymbol{r}$: elementary displacement along *C* (tangent to it)]

$\iint$ is taken over the open surface *S* whose boundary is *C*, i.e. $\partial S = C$ and $d\boldsymbol{S} = \boldsymbol{n} dS$: vector element of area $d\boldsymbol{S} = \boldsymbol{n} dS$ of $\boldsymbol{S}$, with outward unit normal vector $\boldsymbol{n}$, and

$\nabla ... \equiv \boldsymbol{i}\partial ... / \partial x + \boldsymbol{j}\partial ... / \partial y + \boldsymbol{k}\partial ... / \partial z$ is a symbolic operator in rectangular Cartesian coordinates with orthonormal basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ (2)

Hints: Use the following identities, for any three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ (including ∇) ,

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) ; \quad \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{c} \cdot \boldsymbol{a})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \qquad (3)$$

hence, if $\boldsymbol{c}$ = *constant*,

$$\nabla \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{c} \cdot \nabla)\boldsymbol{b} - \boldsymbol{c}(\nabla \cdot \boldsymbol{b}) \qquad (4)$$

Problem 2: Linear algebra

a) Show that the equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx = 1 \qquad (1)$$

can be written into the following matrix form by determining [**M**]:

$$\mathbf{X}^{T}[\mathbf{M}]\mathbf{X} = 1 \text{ where } \mathbf{X} = [x\ y\ z]^{T} \qquad (2)$$

b) For $A = \frac{3}{2}, B = 2, C = \frac{5}{2}, D = -\frac{\sqrt{6}}{3}, E = -\sqrt{2}, F = -\frac{\sqrt{3}}{3}$, find a suitable coordinate transformation [T] such that Eq. (1) can be rewritten as

$$A'x'^2 + B'y'^2 + C'z'^2 = 1 \qquad (3)$$

Determine the transformation matrix and the values of A', B' and C'.

c) Under what condition do Eq. (3) and hence Eq. (1) describe an ellipsoid in a three dimensional space?

Problem 3. Partial differential equation

The governing equation for the vibration of a string of length l (stretched along the x-axis) is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where c is the wave speed.

The boundary conditions are : $u(0,t) = 0$ and $u(l,t) = 0$

Suppose the string is put into an initial static configuration described by $u(x,0) = f(x)$ where $f(x)$ is some specified function.

At t=0 the string is released.

Determine the resulting motion of the string.

Problem 4: **Numerical Analysis**

1) Find the root(s), if it/they exist, of $x = \frac{\sin x}{x}$ within 10^{-3} using Newton-Raphson's method. Discuss the error associated with Newton-Raphson's root-finding method and, in particular, explain what is meant by "quadratic convergence".

2) Gauss-Seidel's method to find the root of $f(x) = 0$ consists in writing $f(x) = x - g(x) = 0$. Choose the same initial guess for *x* as for 1) and proceed iteratively toward the root. Compare the rates of convergence between the Newton-Raphson's method and Gauss-Seidel's method.

3) Consider now the system of equations

$$u(x, y) = 0 \quad \text{and} \quad v(x, y) = 0$$

where *u* and *v* are two known functions (continuous and differentiable). Using 2D-Taylor series expansions for *u* and *v* around (x_i, y_i), derive an expression for an iterative numerical scheme that gives:

x_{i+1} in terms of x_i, u_i, v_i, and the derivatives, $\left(\frac{\partial u}{\partial x}\right), \left(\frac{\partial u}{\partial y}\right), \left(\frac{\partial v}{\partial x}\right), \left(\frac{\partial v}{\partial y}\right)$ and

y_{i+1} in terms of y_i, u_i, v_i, and the derivatives $\left(\frac{\partial u}{\partial x}\right), \left(\frac{\partial u}{\partial y}\right), \left(\frac{\partial v}{\partial x}\right), \left(\frac{\partial v}{\partial y}\right)$.

Discuss how it relates to the standard Newton-Raphson method discussed in 1).