(1) Solve the initial value problem:

$$\frac{dx}{dt} = 2x - 6y,\ x(0) = 2$$

$$\frac{dy}{dt} = x + 7y,\ y(0) = 1$$

(2) Consider the following function:

$$f(x) = \frac{\pi - x}{2}$$

Express $f(x)$ as a Fourier series $S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\lambda_n x) + \sum_{n=1}^{\infty} b_n \sin(\lambda_n x)$ on the interval $(0, 2\pi)$ and answer the following questions:

1. What is the value of the Fourier coefficient a_0?
2. What are the values of the Fourier coefficients $a_n, \quad n = 1, 2, 3...$?
3. What are the values of the Fourier coefficients $b_n, \quad n = 1, 2, 3...$?
4. Does the Fourier series expansion $S(x)$ correctly predicts $f(x)$ at $x = 0$?
5. Does the Fourier series expansion $S(x)$ correctly predicts $f(x)$ at $x = \pi$?
6. Does the Fourier series expansion $S(x)$ correctly predicts $f(x)$ at $x = 2\pi$?
7. Draw on the same schematic $f(x)$ and $S(x)$ for $-\infty < x < \infty$?

(3) A matrix can be used to carry out a linear transformation, which is very useful in many engineering problems. Consider $Ax = y$. Here A is a square matrix and x and y are vectors. Their dimensions are such that this equation makes sense. We can interpret this equation as A maps x to y. The characteristics of the mapping are all contained in A. Therefore it is possible that by investigating them, one is able to determine A.

Consider a circle $x_1^2 + x_2^2 = 1$.

(a) Design a matrix A such that the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ on the circle is mapped to $(\frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}})$ and the point $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is mapped to $(-\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}})$.

(b) Suppose this matrix is used to map all points on the circle. What is the result (show the mathematical representation and a simple sketch)?

(c) Suppose we wish to design another matrix B so that the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ on the circle is mapped to $(\frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}})$ and the point $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is mapped to $(0, 0)$. If this matrix is used to map all points on the circle, what is the result (show the mathematical representation and a simple sketch)?

(d) How many points on the circle are mapped to the single position (0,0)?

(4) A section S of the plane $x + y + \sqrt{2}\, z = 4$ forms the inclined surface of the tetrahedron shown to the right. The volume flux Q of a vector field $\underline{v}$ through the surface S is given by $Q = \int_S v \cdot \underline{n}\, dS$, where $\underline{n}$ is an outward-pointing unit normal vector to the surface S. For the vector field $\underline{v} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$, determine Q two different ways: a) directly from the surface integral for the flux, and b) using the divergence theorem.