GEORGIA INSTITUTE OF TECHNOLOGY

The George W. Woodruff
School of Mechanical Engineering

Ph.D. Qualifiers Exam - Spring Quarter 1998

Applied Mathematics
EXAM AREA

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Assigned Number **(DO NOT SIGN YOUR NAME)**

- **Please sign your name on the back of this page—**

Problem 1

For the following 3D scalar function

$$\phi = \frac{1}{r}$$

where the polar coordinate r is related to the Cartesian coordinates by

$$r = \sqrt{x^2 + y^2 + z^2}$$

(a) Calculate its gradient ($\nabla\phi$) and LaPlacian ($\Delta\phi$).

(b) Calculate the following loop integral on the surface of an unit sphere around the origin directly and using the divergent theorem.

$$\oiint_{r=1} \nabla\phi \cdot d\mathbf{S}$$

(c) Explain why the results calculated from the above two methods are the same or different.

Problem 2

Consider the system of simultaneous equations given by **Ax=b**, where **A** is an $m \times n$ matrix.

a) Under what condition does the above equation has *at least* one solution for *any* **b**?

b) Find the *minimum-norm* solution to **Ax=b** (i.e., the solution **x** that has the smallest magnitude, $\|\mathbf{x}\|$, among all possible solutions) if the condition in **(a)** is satisfied. Express your answer in terms of **A** and **b**.

c) Suppose **Ax=b** does not have a solution for some **b**. Find **x** that minimizes $f(x) = (\mathbf{A}x - \mathbf{b})^T \mathbf{W}(\mathbf{A}x - \mathbf{b})$, where **W** is a symmetric positive definite (weighting) matrix of appropriate dimensions. State the conditions needed for your solution to be unique. Express your answer in terms of **A**, **b**, and **W**.

d) Solve part **(c)** for $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & \cdots & -1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix}^T$, identity weighting matrix **W**, and an arbitrary $\mathbf{b} = [b_1 \quad \cdots \quad b_m]^T$.

Problem 3

A beam whose stiffness is EI rests on an elastic foundation whose stiffness per unit length is k. The beam is loaded at its ends $x = 0$ and $x = L$ by a compressive force P. The differential equation and boundary conditions governing the static displacement of this beam under a certain distributed load are

$$EI\frac{\partial^4 w}{\partial x^4} + P\frac{\partial^2 w}{\partial x^2} + kw = f\frac{x^2}{L^2}$$

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = 0 \text{ and } x = L$$

1. Determine the homogeneous solution of the differential equation in the case where $k = P^2/(8EI)$.

2. For the case where $k = P^2/(8EI)$, determine the solution for w satisfying the boundary conditions.

3. The nature of the homogeneous solution of the differential equation changes if kEI/P^2 exceeds a certain value. What is this critical value?

4. Suppose that kEI/P^2 is twice the critical value established in part (3). What is the homogeneous solution of the differential equation in that case?

Problem 4

Using centered differences in space and time, one can approximate the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

by the following partial difference equation

$$\frac{u_j^{(m+1)} - 2u_j^{(m)} + u_j^{(m-1)}}{(\Delta t)^2} = c^2 \frac{u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}}{(\Delta x)^2}, \quad j, m = 0, 1, 2, \cdots$$

where, for convenience,

$$u_j^{(m)} \equiv u(x_j, t_m), \; x_j = j\Delta x, \; t_m = m\Delta t$$

(1) Please show that the truncation error of the difference equation is $O(\Delta x)^2 + O(\Delta t)^2$.

(2) Assuming that the initial conditions are given by

$$u(x,0) = f(x), \; \frac{\partial u(x,0)}{\partial t} = g(x).$$

Please describe a method to solve the difference equation by marching forward in time.

(3) Please show that a time step

$$\Delta t < \frac{\Delta x}{c}$$

will ensure a stable numerical solution to the difference equation.

Problem 5

Find the solution, $T(x,t)$, of the following problem. Use the method of Laplace Transforms to deal with the time variable t.

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \qquad x \geq 0,\ t \geq 0$$

$$T(x,0) = 0$$

$$\text{at } x = 0, \qquad -k\frac{\partial T}{\partial x} = E\delta(t)$$

$$\text{as } x \to \infty, \qquad \frac{\partial T}{\partial x} \to 0,$$

where $\delta(t)$ is the Dirac delta function. A small table of Laplace Transforms is provided below.

LAPLACE TRANSFORMS

$f(s)$		$F(t)$
$\frac{1}{\sqrt{s}} e^{-\frac{k}{s}}$		$\frac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$
$\frac{1}{\sqrt{s}} e^{\frac{k}{s}}$		$\frac{1}{\sqrt{\pi t}} \cosh 2\sqrt{kt}$
$\frac{1}{s^{3/2}} e^{-\frac{k}{s}}$		$\frac{1}{\sqrt{\pi k}} \sin 2\sqrt{kt}$
$\frac{1}{s^{3/2}} e^{\frac{k}{s}}$		$\frac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$
$e^{-k\sqrt{s}}$	$(k>0)$	$\frac{k}{2\sqrt{\pi t^3}} \exp\left(-\frac{k^2}{4t}\right)$
$\frac{1}{s} e^{-k\sqrt{s}}$	$(k \geq 0)$	$\operatorname{erfc} \frac{k}{2\sqrt{t}}$
$\frac{1}{\sqrt{s}} e^{-k\sqrt{s}}$	$(k \geq 0)$	$\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{k^2}{4t}\right)$
$\frac{1}{s^{\frac{3}{2}}} e^{-k\sqrt{s}}$	$(k \geq 0)$	$2\sqrt{\frac{t}{\pi}} \exp\left(-\frac{k^2}{4t}\right) - k \operatorname{erfc} \frac{k}{2\sqrt{t}} = 2\sqrt{t}\ \mathrm{i}\operatorname{erfc} \frac{k}{2\sqrt{t}}$
$\frac{1}{s^{1+\frac{1}{2}n}} e^{-k\sqrt{s}}$	$(n=0,1,2,\ldots;\ k \geq 0)$	$(4t)^{\frac{1}{2}n}\ \mathrm{i}^n \operatorname{erfc} \frac{k}{2\sqrt{t}}$
$s^{\frac{n-1}{2}} e^{-k\sqrt{s}}$	$(n=0,1,2,\ldots;\ k>0)$	$\frac{\exp\left(-\frac{k^2}{4t}\right)}{2^n\sqrt{\pi t^{n+1}}} H_n\left(\frac{k}{2\sqrt{t}}\right)$