GEORGIA INSTITUTE OF TECHNOLOGY

The George W. Woodruff School of Mechanical Engineering

Ph.D. Qualifiers Exam - Spring Quarter 1997

Applied Mathematics
EXAM AREA

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Assigned Number (DO NOT SIGN YOUR NAME)

- Please sign your name on the back of this page—

Please **print** your name here.

The Exam Committee will get a copy of this exam and will not be notified whose paper it is until it is graded.

Problem 1

The response of a system is governed by the following pair of differential equations:

$$\ddot{x} + 2\dot{x} - 2\dot{y} - 4x = 0$$
$$\dot{y} + 5x - y = \exp(-t)$$

The solution of these equations is real.

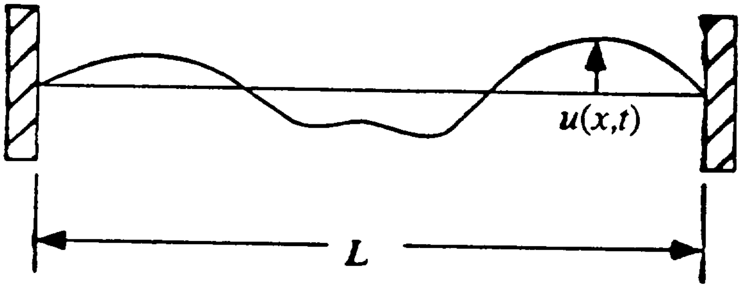
Part (a) Determine the most general ***real*** complementary solution of these differential equations.

Part (b) What is the basic form of the particular solution? Identify the mathematical terms, but do not evaluate the associated coefficients.

Part (c) Consider the case where these equations are associated with an initial value problem. What initial values at $t = 0$ must be specified in order to determine the coefficients of the complementary solution?

Problem 2

Consider the vibration of a string which is both displaced initially from its equilibrium position and released from this position with a specified velocity.



The mathematical problem for this situation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$
$$u(0,t) = 0,$$
$$u(L,t) = 0,$$
$$u(x,0) = f(x),$$
$$\frac{\partial u(x,0)}{\partial t} = g(x),$$

where c, the phase speed, is the square root of the ratio of the tension in the string to its mass per unit length and f and g are specified functions of x, the distance along the string.

a. Solve this problem using the method of *separation of variables.*

b. Specifically determine all coefficients for the case when the string is released from an initial position given by

$$f(x) = \begin{cases} \alpha x, & 0 \le x \le L/2, \\ \alpha(L-x), & L/2 < x \le L \end{cases}$$

with zero initial velocity.

Problem 3

Consider the matrix equation **A**x=b, where **A** is a 3x2 matrix (more equations than unknowns).
a) What is meant by the column space of the matrix **A**?
b) What is the general condition on the vector b for a solution to this system to exist? Discuss the geometric meaning of this condition. Write it in terms of a vector relation.

c) Consider $\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ b_3 \end{bmatrix}$. Find the value of b_3 for a solution to exist. Then, find the solution.
d) Find the rank of **A**.

Problem 4.

Let A be a *symmetric* $n \times n$ matrix with real elements. Knowing that an $n \times n$ symmetric matrix always has n linearly independent eigenvectors prove that

a) All the eigenvalues of A are real.

b) There exists an orthonormal set of eigenvectors $\{v_1, \dots v_n\}$.

c) Find all the eigenvalues and an *orthonormal* set of eigenvectors for

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

d) Provide a simple *iterative* numerical routine, which does not involve any root finding, for approximating the *largest* eigenvalue and the corresponding eigenvector of A. Justify the convergence of your routine and illustrate it on matrix A in part (c).

Optional: Do part (e) of this problem for extra credit if you have sufficient time.

e) Suppose matrix A has n *distinct* eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$. Show that λ_2 is the largest eigenvalue of $A_1 = A - \lambda_1 v_1 v_1^T$. Use this fact to extend your algorithm in (d) to find all the eigenvalues of A.