Problem 1:

a) Solve the following equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x} \quad ;$$

with

$$y(1) = 1, \;\; y'(1) = 1.$$

b) If the equation were instead given as

$$x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0,$$

what method would you use to solve it for a solution near $x = 0$ and why?

Problem 2:

Let's consider a system of n ordinary differential equations

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A} \cdot \mathbf{x}(t), \qquad (1)$$

where $\mathbf{x}(t)$ is a column vector consisting of n unknowns, $x_1(t)$, $x_2(t)$, ... $x_n(t)$, and $\mathbf{A}$ is a $n \times n$ constant matrix. Let's further assume that $\mathbf{A}$ is real and symmetric.

(a). Let $\mathbf{v}_m$ ($m = 1, 2, \ldots n$) be the eigenvectors of $\mathbf{A}$ and λ_m be their corresponding eigenvalues. Please find the general solution to (1) in terms of $\mathbf{v}_m$ and λ_m.

(b). If the initial condition for the above problem is given by

$$\mathbf{x}(0) = \mathbf{b}, \qquad (2)$$

where $\mathbf{b}$ is a given $n \times 1$ constant vector, what is the solution to (1) that also satisfies the initial condition (2)?

Problem 3:

Consider the vector field $\mathbf{F}=yz\mathbf{i}+(1-x)y\mathbf{j}+(1+xz)\mathbf{k}$ in the 3-dimensional xyz space with unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ along x, y, and z, respectively.

a) Is it possible to find a potential function $\varphi(x,y,z)$ such that $\nabla\varphi=\mathbf{F}$? why?

b) Evaluate the surface integral $\int_S \mathbf{F}\bullet\vec{n}\,dS$ where S is the surface of the upper unit hemisphere $S=\{(x,y,z):x^2+y^2+z^2=1,\ z>0\}$ and $\vec{n}$ is the unit vector normal to S.

c) Evaluate the integral $\oint_C \mathbf{F}\bullet(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$, where C is the boundary of the unit circle centered at (0,1,0): $C=\{(x,y,z): x^2+(y-1)^2=1,\ z=0\}$.

d) Evaluate $\int_S (\nabla\times\mathbf{F})\bullet\vec{n}\,dS$ where S is an arbitrary surface in the region $D=\{(x,y,z): z\geq 0\}$ bounded by curve C in part (c): $C=\{(x,y,z): x^2+(y-1)^2=1,\ z=0\}$ and $\vec{n}$ is the unit vector normal to S.

Problem 4: This is a question about the Fourier transform.

1) Consider a phase modulated signal $x(t) = \exp(i\,\Phi(t))$, where $\Phi(t)$ is a continuous and differentiable function of time (t) called the "instantaneous phase". The "instantaneous frequency $f_i(t)$" is defined as

$$f_i(t) = \frac{1}{2\pi}\frac{d\Phi(t)}{dt} \qquad \text{Eq. (1)}$$

Compute $f_i(t)$ for the following two cases

- a) $\Phi(t) = 2\pi f_c t$ (i.e. a sinusoidal signal) or
- b) $\Phi(t) = 2\pi(f_c t + \alpha t^2/2)$ (i.e. a "Chirp" signal or "Linear Frequency Modulated" signal).

Describe briefly the time dependency of the "instantaneous frequency $f_i(t)$"

2) For arbitrary time-domain function $x(t)$, and a given time t, the "delay function" $R(\tau;t)$ is defined as:

$$R(\tau;t) = x(t+\tau/2)x(t-\tau/2) \qquad \text{Eq. (2)}$$

For a fixed value of the absolute time t, the function $R(\tau;t)$ is only a function of the time-delay τ. The **Fourier transform** $W(f;t)$ of the delay function $R(\tau;t)$ (For a fixed value of t,) is given, as usual, by

$$W(f;t) = \int_{-\infty}^{\infty} R(\tau;t)\exp(-i2\pi f\tau)d\tau \qquad \text{Eq. (3)}$$

Hence the absolute time t can be viewed as a parameter that can change arbitrarily.

a. For $x(t) = \cos(2\pi f_c t)$. **Compute** the corresponding "delay function" $R(\tau;t)$ and associated Fourier transform $W(f;t)$ (For a fixed value of t);
b. How does this relate to what you found in 1.a ?

3) If $X(f)$ is the Fourier transform of the function x(t), **show** that function $W(f;t)$ can also be computed from the following **Fourier transform** (for a fixed value of f):

$$W(f;t) = \int_{-\infty}^{\infty} X(f+\upsilon/2)X^*(f-\upsilon/2)\exp(+i2\pi\upsilon\, t)d\upsilon, \qquad \text{Eq. (4)}$$

where the symbol * denotes complex conjugation.

Hint: Represent $X(f+\upsilon/2)X^*(f-\upsilon/2)$ in terms of x(t) by using the definition of the Fourier transform.