Problem 1: Consider the system of simultaneous equations given by **Ax**=**b**, where **A** is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 2 \\ 0 & 5 & 1 & -4 \end{bmatrix}$$

a) Does this system have a solution for an arbitrary **b**? why?
b) Find all possible unit vectors **n** that satisfy **An**=0.
c) Find all possible **b**'s for which **Ax**=**b** has at least one solution.
d) Express all possible solutions to **Ax**=**b** for **b**=$[1\ 1\ 1\ 1]^T$.
e) Choose **x** from all possible solutions in (d) that has the smallest magnitude or equivalently the one that minimizes $\mathbf{x}^T\mathbf{x}$.

Problem 2: Classify the equation (i.e. what type) and find analytical solution of the following problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 6 \quad (t > 0; -\infty < x < \infty)$$

$$u(x, t = 0) = x^2 \qquad \text{Eq. (1)}$$

$$\frac{\partial u}{\partial t}(x, t = 0) = 2x$$

Hint: a <u>general</u> solution of $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (t > 0; -\infty < x < \infty)$ can be expressed as a superposition of two arbitrary functions along the characteristic lines of the governing equation, i.e., $u(x,t) = f(x - at) + g(x + at)$

Problem 3: Consider the transient 1-D heat conduction in a rod whose length L = 5 cm and thermal diffusivity k = 0.80 cm^2/s, which can be described using the following equation

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

supplied with the following boundary and initial conditions

$T(x = 0, t) = 200\ ^{o}$C
$T(x = L, t) = 50\ ^{o}$C
$T(x, t=0) = 0\ ^{o}$C

Two methods for numerically solving this equation are given by

Method (1): $$\frac{T_i^{n+1} - T_i^n}{\Delta t} = k \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2}$$

Method (2): $$\frac{T_i^{n+1} - T_i^n}{\Delta t} = k \frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{(\Delta x)^2}$$

a. Restate Equation/Method (1) in a form where it can easily be used to find numerically the temperature distribution in the bar. It is desired to find the temperature at increments of 1 cm, determine the time step required to insure that the solution does not oscillate.

b. Restate Equation/Method (2) in a form where it can easily be used to find numerically the temperature distribution in the bar. It is desired to find the temperature at increments of 1 cm. Set up in matrix form, the required linear algebraic expressions used to solve for the temperature distribution in the rod. Use the numerical values given and the same time step as used in part a. Describe the solution method.

c. Will Method (2) yield a stable solution if a Δt larger than that found in part (a) is used?

Problem 4: In fluid dynamics, a "Beltrami flow" is one for which the vorticity field $\boldsymbol{\omega}$ (defined by $\boldsymbol{\omega} = \nabla \times \mathbf{V}$, where $\mathbf{V}$ is the velocity vector) is *parallel* to the velocity field $\mathbf{V}$. Consider a steady flow field in Cartesian coordinates parallel to a plane boundary given by $\mathbf{V} = \left(u(z), v(z), 0\right)$.

a. Devise *non-constant* velocity components that will make this a Beltrami flow.

b. What is the vorticity field corresponding to your velocity?