(1) Please solve the following equation:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 65\sin(\omega t)$$

Given $\omega = 2$ and

$$\text{at } t = 0: \ x = 0; \frac{dx}{dt} = 2$$

(2) Consider the following equation

$$\mathbf{x}^T \mathbf{A}\mathbf{x} = 128$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

This equation represents an ellipse on the (x_1, x_2) plane, with the origin as its center.

[a] Find the points on the ellipse which are the farthest and closest from the origin of the plane.

[b] Design an ellipse, with the origin as the center, whose *farthest* points from the origin are the same as the *closest* points in [a]. Represent this ellipse in terms of (x_1, x_2).

(Hint: consider the geometric interpretation of the eigenvalues of a matrix)

(3) Let $f(x)=|x|$ be defined on $[-\pi,\pi]$. Its Fourier series may be written as

$$f(x)=\lim_{N\to\infty} f_N(x)$$

where

$$f_N(x)=a_0+\sum_{n=1}^{N}\left(a_n\cos nx+b_n\sin nx\right)$$

(a) Find all the coefficients a_0, a_n and b_n.

(b) For very large *N*, investigate the behavior of the least square error

$$\sigma_N=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x)-f_N(x)\right]^2 dx$$

Hint: you may need to know that for very large *N*

$$\sum_{n=2N+1}^{\infty}\frac{1}{(2n-1)^4}\approx\int_N^{\infty}\frac{dx}{(2x-1)^4}$$

(4) A temperature sensor travels on a helical path through a temperature field $T(x, y, z)$. The helical path is given by the position vector

$$\underline{R}(t) = x(t)\,\underline{i} + y(t)\,\underline{j} + z(t)\,\underline{k} = \cos(t)\,\underline{i} + \sin(t)\,\underline{j} + \tfrac{\sqrt{15}}{7}\,t\,\underline{k}, \quad t \geq 0,$$

where t is time. The temperature recorded by the sensor varies with the arclength s of the helix as $T = T_0 + \Delta T \sin(s)$, where T_0 and ΔT are constants.

a) Determine the vector velocity of the sensor.

b) Determine the material derivative of the temperature following the sensor.

c) If $\nabla T \times \underline{k} = 0$, find the functional form of $T(x, y, z)$.

d) What is the wavelength of the temperature variation in the z-direction?