M.E. Ph.D. Qualifier Exam
Fall Semester 1999

GEORGIA INSTITUTE OF TECHNOLOGY

The George W. Woodruff
School of Mechanical Engineering

Ph.D. Qualifiers Exam - Fall Semester 1999

Applied Mathematics
EXAM AREA

Assigned Number **(DO NOT SIGN YOUR NAME)**

- **Please sign your name on the back of this page—**

Applied Mathematics Qualifying Examination
Fall 1999

Answer any **five** of the following six questions. Be sure to answer all parts of each question you select.

1. Consider the following two-dimensional eigenvalue problem in Cartesian Coordinates:

$$\nabla^2 p_{nm} + \alpha_{nm}^2 p_{nm} = 0,$$

Where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad n, m = 1, 2, \cdots$$

$$p_{nm}(0, y) = p_{nm}(a, y) = p_{nm}(x, 0) = p_{nm}(x, b) = 0$$

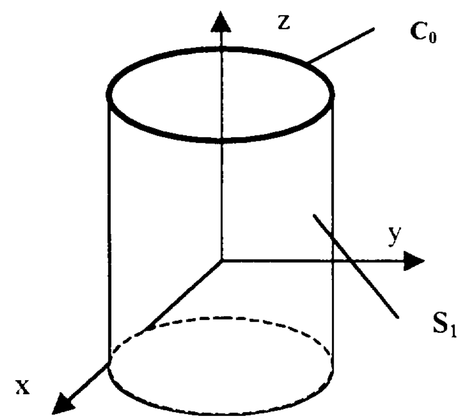
Find the eigenvalues α_{nm}, the eigenfunctions $p_{nm}(x,y)$, and establish the orthogonality property (but no need to solve it) of the eigenfunctions.

2. Consider the following vector field:

$$\mathbf{v}(x,y,z) = y\mathbf{i} - x\mathbf{j} + (z^2-1)\mathbf{k}$$

where **i, j, k** are unit vectors along the Cartesian x,y,z axes, respectively.

Consider the region bounded by the planes $z = \pm 4$ and the cylinder $x^2 + y^2 = 5$.



a) Evaluate the integral $\oint_{C_0} \mathbf{v} \cdot d\mathbf{r}$ on the edge C_0.

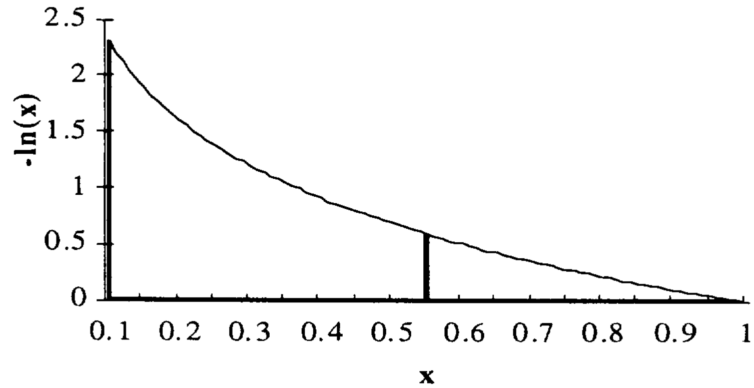
$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

b) Evaluate the integral $\iint_{S_1} \mathbf{n} \cdot \mathbf{v} dA$ on the cylindrical surface S_1.

c) Can a nontrivial ($\mu \neq 0$) scalar function $\mu(x,y,z)$ exist such that the vector field $\mathbf{w} = \mu\mathbf{v}$ is irrotational? Why or why not?

d) Could **v** represent the steady flow of a fluid through this region? Why or why not?

3. The function f(x) = - ln (x)
(where ln represents the natural logarithm)



can be used to generate the following table of data:

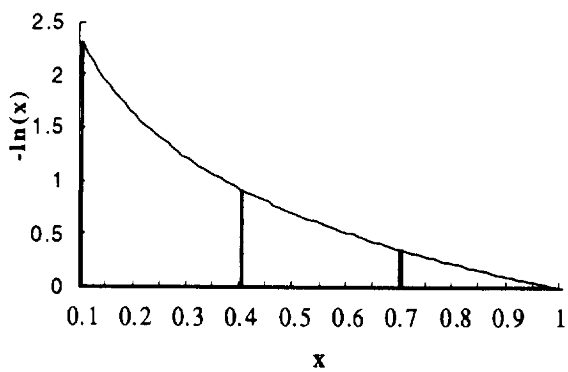
x	f(x)
0.10	2.3026
0.55	0.5978
1.00	0.0000

(a) Evaluate the integral from a=0.1 to b=1.0 analytically

(b) Evaluate this integral using the trapezoidal rule and the data in the table above.

(c) Evaluate this integral using Simpson's 1/3 rule and the data in the table above.

(d) Assume that the area is divided into three intervals. Use Simpson's 3/8 rule to compute the value of the integral from the data in the table.



x	f(x)
0.10	2.303
0.40	0.9163
0.70	0.3567
1.00	0.0000

(e) Comment on your answers.

4. Consider the following boundary value problem.

$$-T\frac{d^2u}{dx^2}=f(x), \quad 0<x<L$$

$$u(0) = 0, \ u(L) = 0$$

a) Derive the Green's function for this problem.

b) Construct the solution to this boundary value problem using the Green's function.

c) What physical problem can be solved by this boundary value problem? (optional)

5. Given the tensor **A** with matrix of components

$$[A] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & \xi \\ 4\sqrt{6} & \xi & 2 \end{bmatrix}$$

(a) Decompose **[A]** into symmetric and skew symmetric parts, leaving ξ as a variable.

(b) Determine the value of ξ for which at least one of the three eigenvalues of the symmetric part $[\mathbf{A}]_{sym}$ is equal to 2. For this value of ξ, determine the other two eigenvalues as well.

Are the eigenvalues real? Why or why not?

(c) Determine the unit eigenvectors of the symmetric part of **A** for the value of ξ found in part (b)

(d) Is this set of eigenvectors linearly independent?

(e) Is **[A]** orthogonal? Why or why not?

6. Given the differential equation $\dot{y} + 2y = 5\sin t$ with initial condition $y(0) = 1$, use Laplace transforms (a table is provided) to answer the following questions:

   a. Determine $Y(s)$, the Laplace transform of $y(t)$.

   b. Solve for $y(t)$.

   c. Sketch the transient part of $y(t)$.

   d. Sketch the steady state part of $y(t)$

   e. Apply the final value theorem and explain the result.

Table 6-1. Laplace transform pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!}\quad (n = 1, 2, 3, \ldots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!}t^{n-1}e^{-at}\quad (n = 1, 2, 3, \ldots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at}\quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$

Table 6-1. (Continued)

	$f(t)$	$F(s)$
17	$\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$
18	$\frac{1}{a^2}(1-e^{-at}-ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at-1+e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
21	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin \omega_n\sqrt{1-\zeta^2}\,t$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t-\phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
24	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t+\phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
25	$1-\cos \omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
26	$\omega t-\sin \omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
27	$\sin \omega t-\omega t\cos \omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
28	$\frac{1}{2\omega}t\sin \omega t$	$\frac{s}{(s^2+\omega^2)^2}$
29	$t\cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
30	$\frac{1}{\omega_2^2-\omega_1^2}(\cos \omega_1 t-\cos \omega_2 t)\quad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
31	$\frac{1}{2\omega}(\sin \omega t+\omega t\cos \omega t)$	$\frac{s^2}{(s^2+\omega^2)^2}$

Table 6-2. Properties of Laplace transforms

1	$\mathcal{L}[Af(t)] = AF(s)$
2	$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathcal{L}_\pm\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathcal{L}_\pm\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathcal{L}_\pm\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}\overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathcal{L}_\pm\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{\left[\int f(t)\,dt\right]_{t=0\pm}}{s}$
7	$\mathcal{L}_\pm\left[\iint f(t)\,dt\,dt\right] = \frac{F(s)}{s^2} + \frac{\left[\int f(t)\,dt\right]_{t=0\pm}}{s^2} + \frac{\left[\iint f(t)\,dt\,dt\right]_{t=0\pm}}{s}$
8	$\mathcal{L}_\pm\left[\int\cdots\int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n}\frac{1}{s^{n-k+1}}\left[\int\cdots\int f(t)(dt)^k\right]_{t=0\pm}$
9	$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
10	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)\quad$ if ?
11	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$
12	$\mathcal{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s)\quad \alpha \geq 0$
13	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
14	$\mathcal{L}[t^2 f(t)] = \frac{d^2}{ds^2}F(s)$
15	$\mathcal{L}[t^n f(t)] = (-1)^n\frac{d^n}{ds^n}F(s)\quad n = 1, 2, 3, \ldots$
16	$\mathcal{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds$
17	$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$