GEORGIA INSTITUTE OF TECHNOLOGY

The George W. Woodruff
School of Mechanical Engineering

Ph.D. Qualifiers Exam - Fall Quarter 1997

Applied Math
EXAM AREA

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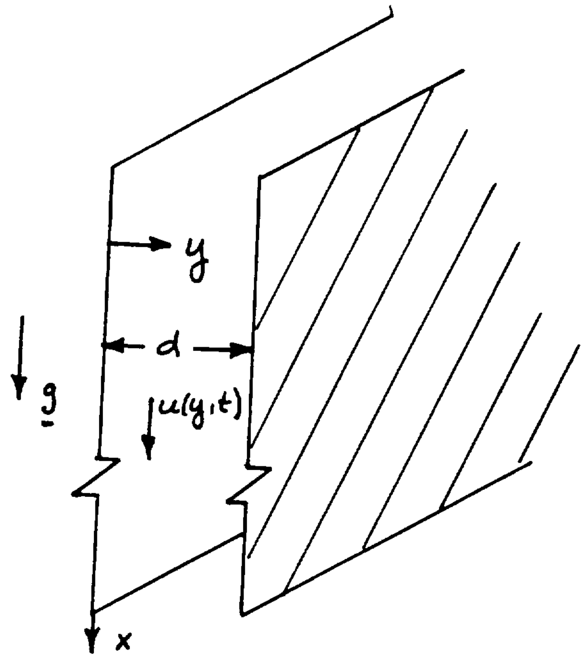
Assigned Number **(DO NOT SIGN YOUR NAME)**

- **Please sign your name on the back of this page—**



1. A liquid is held in place between two infinite, vertically oriented solid walls spaced a distance d apart. At time $t = 0$, the liquid is allowed to flow downward under the action of gravity. The problem governing the velocity $u(y,t)$ of the liquid is:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + g$$
$$u(0,t) = 0$$
$$u(d,t) = 0$$
$$u(y,0) = 0$$

where ν is the liquid's kinematic viscosity and g is the gravitational acceleration constant.

a) **Scale the problem** above using the distance d between the plates as the length scale and the diffusion time d^2/ν as the time scale. **Choose a velocity scale U which most simplifies the PDE.**

b) Solve the dimensionless problem using the method of **separation of variables.**

You may find the following mini table of integrals helpful:

$$\int x \cos ax\, dx = \frac{1}{a^2}\cos ax + \frac{1}{a} x \sin ax$$

$$\int x^2 \cos ax\, dx = \frac{2x}{a^2}\cos ax - \frac{2}{a^3}\sin ax + \frac{x^2}{a}\sin ax$$

$$\int x^3 \cos ax\, dx = \frac{(3a^2x^2 - 6)\cos ax}{a^4} + \frac{(a^2x^2 - 6x)\sin ax}{a^3}$$

$$\int x \sin ax\, dx = \frac{1}{a^2}\sin ax - \frac{1}{a} x \cos ax$$

$$\int x^2 \sin ax\, dx = \frac{2x}{a^2}\sin ax + \frac{2}{a^3}\cos ax - \frac{x^2}{a}\cos ax$$

$$\int x^3 \sin ax\, dx$$
$$= \frac{3x^2}{a^2}\sin ax - \frac{6}{a^4}\sin ax - \frac{x^3}{a}\cos ax + \frac{6x}{a^3}\cos ax$$

2. An $n \times n$ matrix **R** is said to be a *rotation* matrix if it preserves the scalar product of any two vectors, that is, for any two vectors **v** and $\mathbf{w} \in R^n$, we have $<\mathbf{Rv}, \mathbf{Rw}> = <\mathbf{v}, \mathbf{w}>$.

   (a) Prove that the a rotation matrix **R** has the following properties:

   1. It is orthonormal, i.e., $\mathbf{R}^T\mathbf{R}=\mathbf{I}$, where **I** is the identity matrix.
   2. Its determinant is unity (± 1).
   3. The magnitude of each of its eigenvalues is one, i.e., if λ is an eigenvalue of **R**, then $|\lambda| = 1$.

   (b) Find the rotation matrix **R** if $\mathbf{R}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0.6\\0.8\end{bmatrix}$ and $\mathbf{R}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$.

   (c) Let **S** be a skew-symmetric matrix (i.e., $\mathbf{S}^T = -\mathbf{S}$). Prove that the solution to the following matrix ordinary differential equation is a rotation matrix:

$$\frac{d}{dt}\mathbf{R}(t) = \mathbf{S}\mathbf{R}(t)$$

$$\mathbf{R}(0) = \mathbf{I}$$

   (d) If $\mathbf{S} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$ find $\mathbf{R}(t)$ satisfying the above equation.

3. Consider the four vector fields given below:

$$f_{\pm} = xi \pm yj \qquad g_{\pm} = yi \pm xj$$

where x and y are Cartesian coordinates and $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors along the x and y axes, respectively.

(a) Calculate the curl and divergence of these four vector fields.

(b) Use the Stokes theorem to compute the four loop integrals below:

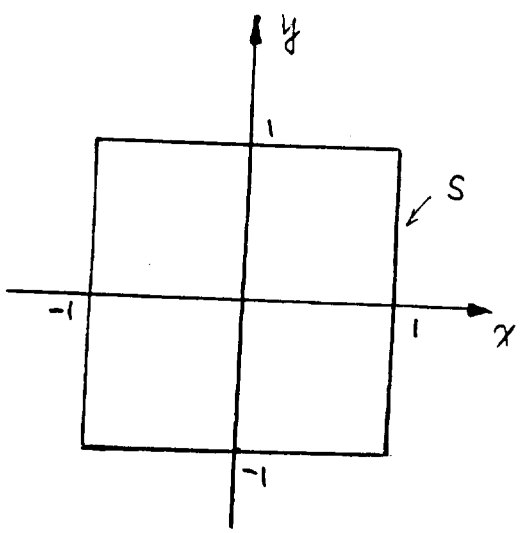
$$I_{f_{\pm}} = \oint_S f_{\pm} \cdot ds \qquad I_{g_{\pm}} = \oint_S g_{\pm} \cdot ds$$

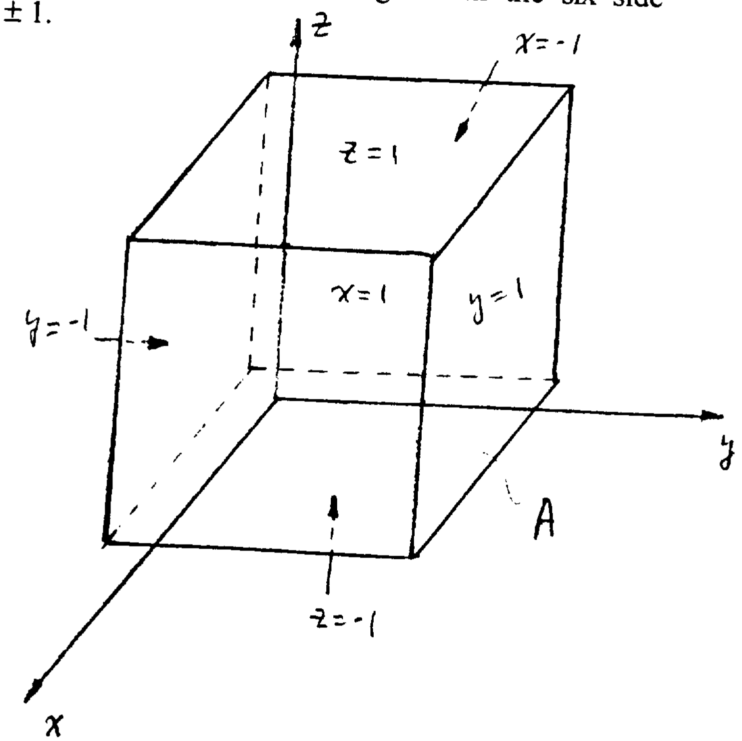
where $\boldsymbol{ds}$ is the line element vector along the contour of integration which is the closed contour of a square around the origin with the four sides being $x = \pm 1$ and $y = \pm 1$.

(c) Use the divergence theorem to compute the four surface integrals below:

$$II_{f_{\pm}} = \iint_A f_{\pm} \cdot dA \qquad II_{g_{\pm}} = \iint_A g_{\pm} \cdot dA$$

where $\boldsymbol{dA}$ is the surface element vector whose direction is normal to the surface of integration which is the closed surface of a cube around the origin with the six side surfaces being $x = \pm 1, y = \pm 1$ and $z = \pm 1$.





4. Consider the following Fredholm integral equation of the second kind for the unknown function $g(x)$,

$$\int_a^b K(x,t)g(t)dt + \lambda g(x) = f(x),$$

where a, b, λ are given constants ($b > a$), and $f(x)$, $K(x,t)$ are known functions. To obtain numerical solutions, apply a simple numerical integration method to the integral term to reduce the equation to a system of linear algebraic equations of the following form

$$(\mathbf{K} + \lambda \mathbf{I})\mathbf{g} = \mathbf{f}$$

where **K**, **I** are $N \times N$ matrices and the unknown vector is

$$\mathbf{g} = \begin{bmatrix} g(x_1) \\ g(x_2) \\ g(x_3) \\ \vdots \\ g(x_N) \end{bmatrix}$$

with $a \le x_1 < x_2 < x_3 < \cdots < x_N \le b$. Derive explicit expressions for all components in **K**, **I** and **f**.